A Graphical Look at Why WGARP Implies GARP For Two-Commodity Choice

Samiran Banerjee and James H. Murphy

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This is a supplement to our paper titled "A Simplified Test for Preference Rationality of Two-Commodity Choice". Its objective is to provide a graphical intuition for the result that a violation of k-wise GARP implies at least one pairwise violation of WGARP when there are only two goods in the commodity space. To illustrate this, consider the simplest instance when there are three observations $S = \{(p^1, x^1), (p^2, x^2), (p^3, x^3)\}$ and where all prices and chosen bundles are distinct. Suppose that 3-wise GARP is violated: $x^1 R^0 x^2$ and $x^2 R^0 x^3$ hold, while $x^3 P^0 x^1$. The inequalities expressed by these revealed preferred relations are

$$[p^1 x^1 \geq p^1 x^2] \ \& \ [p^2 x^2 \geq p^2 x^3] \ \& \ [p^3 x^3 > p^3 x^1].$$

Since we wish to demonstrate that this implies at least one WGARP violation, assume by way of contradiction that WGARP is not violated for any pair of observations from S, i.e., for any pair of observations i and j from S ($i, j = 1, 2, 3$, $i \neq j$),

$$[p^i x^i \geq p^i x^j] \ \& \ [p^j x^j \leq p^j x^i].$$

Ranking the budget lines from steepest to flattest, there are six possibilities: (i) $p^1 > p^2 > p^3$, (ii) $p^1 > p^3 > p^2$, (iii) $p^2 > p^1 > p^3$, (iv) $p^2 > p^3 > p^1$, (v) $p^3 > p^1 > p^2$, and (vi) $p^3 > p^2 > p^1$. We analyze each case below.

Case 1: $p^1 > p^2 > p^3$

In Figure 1, OAD is the budget set from which x^1 is chosen, BE the budget line $p^2 x^1$ and CF the budget line $p^3 x^1$. From $x^3 P^0 x^1$ (i.e., $p^3 x^3 > p^3 x^1$), x^3 lies strictly to the right of CF. However, x^3 cannot lie in Ax^1C because that would result in a WGARP violation between x^1 and x^3. Thus x^3 must lie strictly to the right of Ax^1F. Since $x^1 R^0 x^2$ (i.e., $p^1 x^1 \geq p^1 x^2$), x^2 must lie in OAD; however, x^2 cannot lie in Ax^1B because that would result in a WGARP violation between x^1 and x^2. Thus x^2 must lie in OBx^1D (excluding the segment Bx^1). For any x^2 in OBx^1D (excluding segment Bx^1) and x^3 in the area to the right of Ax^1F, $x^2 R^0 x^3$ (i.e., $p^2 x^2 \geq p^2 x^3$) is not possible, yielding a contradiction.

Case 2: $p^1 > p^3 > p^2$

In Figure 2, OAD is the budget set from which x^1 is chosen, CF the budget line p^2x^1 and BE the budget line p^3x^1. From $x^3P^0x^1$, x^3 lies strictly to the right of BE. However, x^3 cannot lie in Ax^1B because that would result in a WGARP violation between x^1 and x^3. Thus x^3 must lie strictly to the right of Ax^1E.

Since $x^1R^0x^2$, x^2 must lie in OAD; however, x^2 cannot lie in Ax^1C because that would result in a WGARP violation between x^1 and x^2. Thus x^2 must lie in OCx^1D (excluding the segment Cx^1). But because $x^2R^0x^3$, and x^3 lies to the right of Ax^1E, x^2 can be further restricted to lie in the shaded area Cx^1HG (excluding the segment Cx^1) where the line GH is parallel to CF. This is because if x^2 were in $OGHD$ instead, the budget line p^2x^2 would have no points in common with the area to the right of Ax^1E, so $x^2R^0x^3$ would not be possible. For any x^2 in Cx^1HG, x^3 must lie within the budget p^2x^2 (not shown) and in Ex^1F (excluding the segment x^1E). But for any x^3 in this subset of Ex^1F, it must be the case that $x^3P^0x^2$, implying a WGARP violation between x^2 and x^3.

Case 3: $p^2 > p^1 > p^3$

In Figure 3, OBE is the budget set from which x^1 is chosen, AD the budget line p^2x^1 and CF the budget line p^3x^1. From $x^3P^0x^1$, x^3 lies strictly to the right of CF. However, x^3 cannot lie in Bx^1C because that would result in a WGARP violation between x^1 and x^3. Thus x^3 must lie strictly to the right of Bx^1F.

Since $x^1R^0x^2$, x^2 must lie in OBE; however, x^2 cannot lie in Dx^1E because that would result in a WGARP violation between x^1 and x^2. Thus x^2 must lie in OBx^1D (excluding the segment x^1D). But because $x^2R^0x^3$, x^2 must lie in GBx^1D, where BG is parallel to AD. This is because if x^2 were in OBG instead, the budget line p^2x^2 would have no points in common with the area to the right of Bx^1F, so $x^2R^0x^3$ would not be possible. For any x^2 in GBx^1D, x^3 must lie within the budget p^2x^2 (not shown) and in Ax^1B (excluding the segment x^1B). But for any x^3 in this subset of Ax^1B, it must be the case that $x^3P^0x^2$, implying a WGARP violation between x^2 and x^3.

Case 4: $p^2 > p^3 > p^1$

In Figure 4, OCF is the budget set from which x^1 is chosen, AD the budget line p^2x^1 and BE the budget line p^3x^1. From $x^3P^0x^1$, x^3 lies strictly to the right of BE. However, x^3 cannot lie in Ex^1F because that would result in a WGARP violation between x^1 and x^3. Thus x^3 must lie strictly to the right of Bx^1F.

Since $x^1R^0x^2$, x^2 must lie in OCF; however, x^2 cannot lie in Dx^1F because that would result in a WGARP violation between x^1 and x^2. Thus x^2 must lie in OCx^1D (excluding the segment x^1D). But because x^3 lies to the right of Bx^1F and $x^2R^0x^3$, x^2 must lie in the shaded area GHx^1D where BG is parallel to AD. This is because if x^2 were in $OCHG$ instead, the budget line p^2x^2 would have no points in common with the area to the right of Bx^1F, so $x^2R^0x^3$ would not be possible. Because x^3 lies to the right of Bx^1F, for any x^2 in GHx^1D with $x^2R^0x^3$, x^3 must lie within the budget p^2x^2 (not shown) and in Ax^1B (excluding the segment x^1B). But for any x^3 in this subset of Ax^1B, it must be the case that $x^3P^0x^2$, implying a WGARP violation between x^2 and x^3.

Case 5: $p^3 > p^1 > p^2$

In Figure 5, OBE is the budget set from which x^1 is chosen, CF the budget line p^2x^1 and AD the budget line p^3x^1. From $x^3P^0x^1$, x^3 lies strictly to the right of AD. However, x^3 cannot lie in Dx^1E because that would result in a WGARP violation between x^1 and x^3. Thus x^3 must lie strictly to the right of Ax^1E.

Since $x^1R^0x^2$, x^2 must lie in OBE; however, x^2 cannot lie in Bx^1C because that would result in a WGARP violation between x^1 and x^2. Thus x^2 must lie in OCx^1E (excluding the segment Cx^1). But because $x^2R^0x^3$, x^2 must lie in GCx^1E, where GE is parallel to CF. This is because if x^2 were in OGE instead, the budget line p^2x^2 would have no points in common with the area to the right of Ax^1E, so $x^2R^0x^3$ would not be possible. For any x^2 in GCx^1E, x^3 must lie within the budget p^2x^2 (not shown) and in Ex^1F. But for any x^3 in this subset of Ex^1F, it must be the case that $x^3P^0x^2$, implying a WGARP violation between x^2 and x^3.

Case 6: $p^3 > p^2 > p^1$

In Figure 6, OCF is the budget set from which x^1 is chosen, BE the budget line p^2x^1 and AD the budget line p^3x^1. From $x^3P^0x^1$, x^3 lies strictly to the right of AD. However, x^3 cannot lie in Dx^1F because that would result in a WGARP violation between x^1 and x^3. Thus x^3 must lie strictly to the right of Ax^1F.

Since $x^1R^0x^2$, x^2 must lie in OCF; however, x^2 cannot lie in Ex^1F because that would result in a WGARP violation between x^1 and x^2. Thus x^2 must lie in OCx^1E (excluding the segment x^1E). For any x^2 in OCx^1E and x^3 in the area to the right of Ax^1F, $x^2R^0x^3$ is not possible, yielding a contradiction.

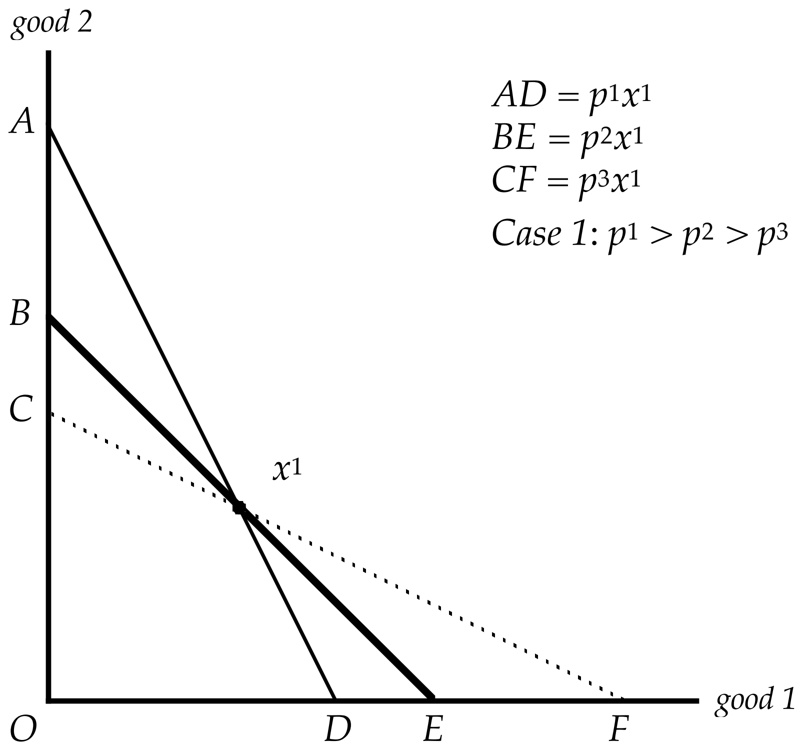

Figure 1

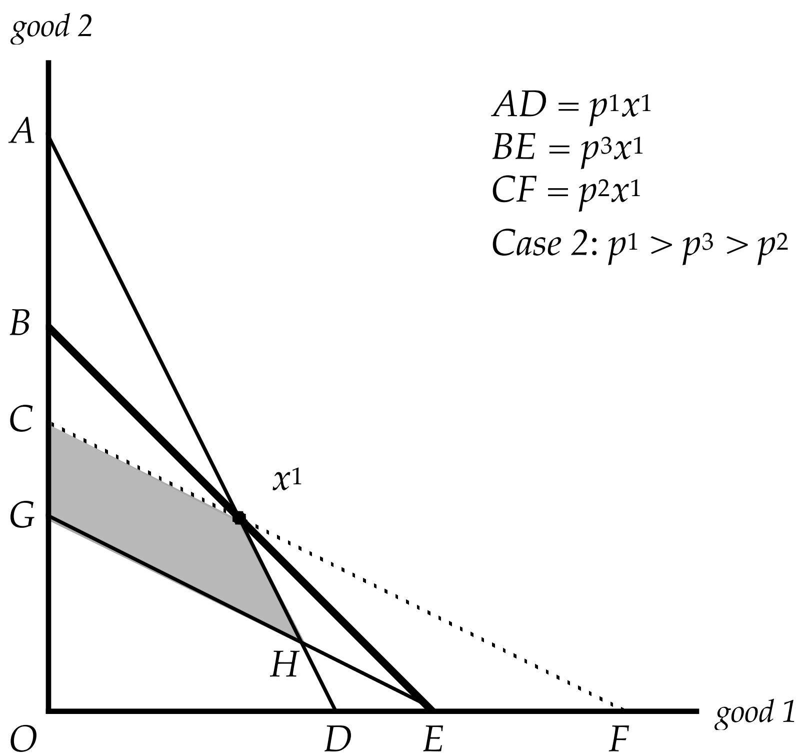

Figure 2

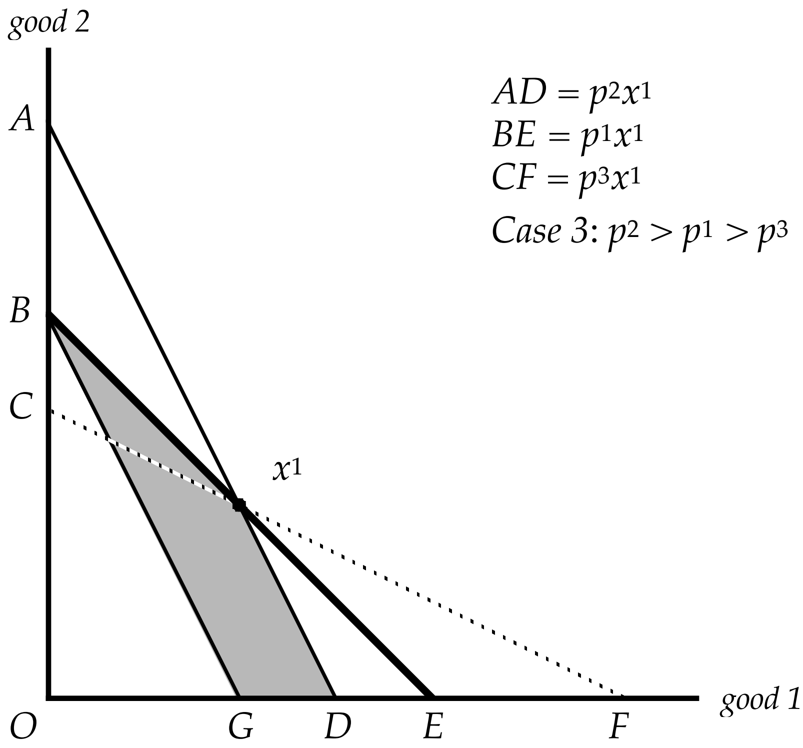

Figure 3

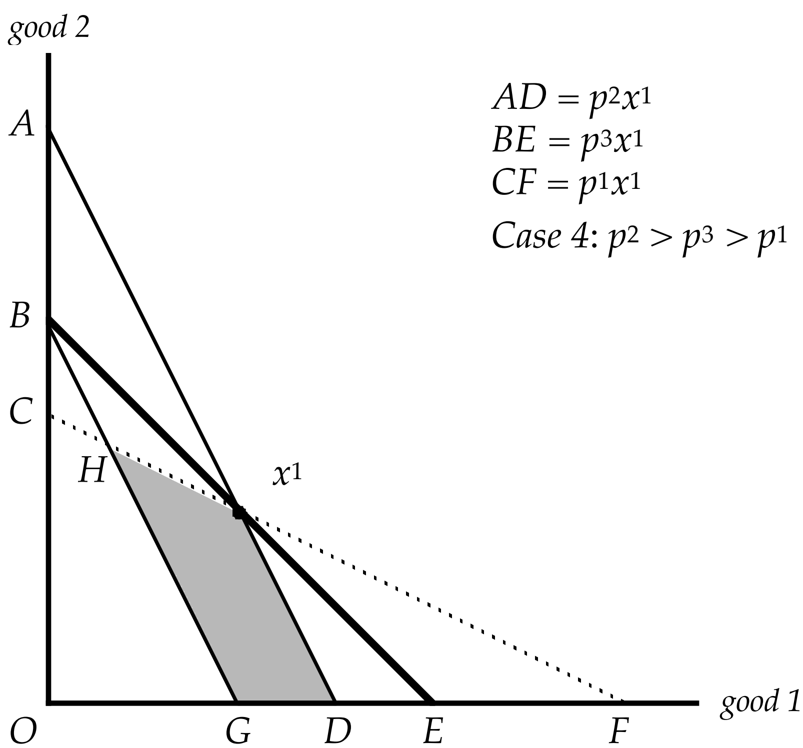

Figure 4

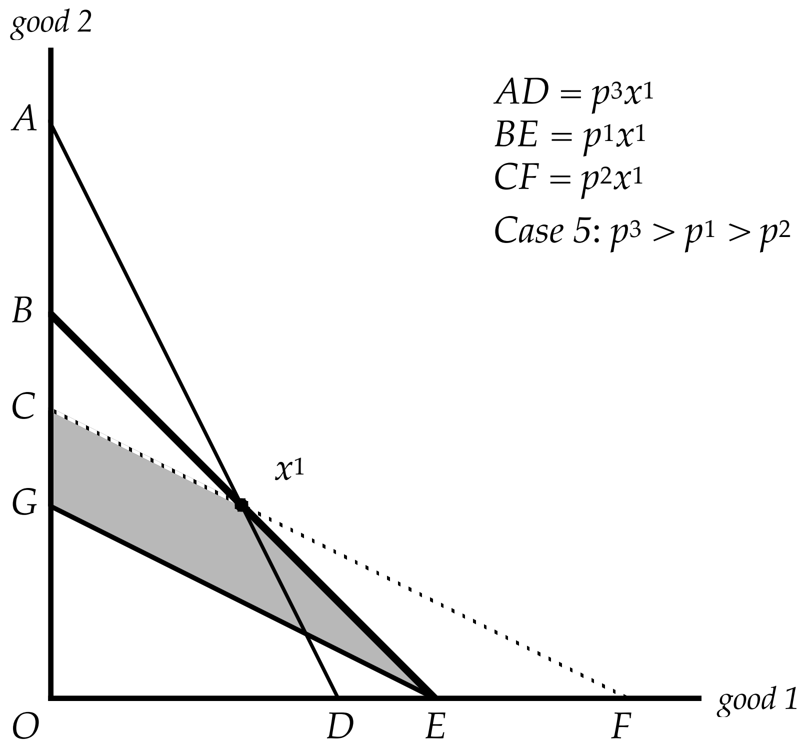

Figure 5

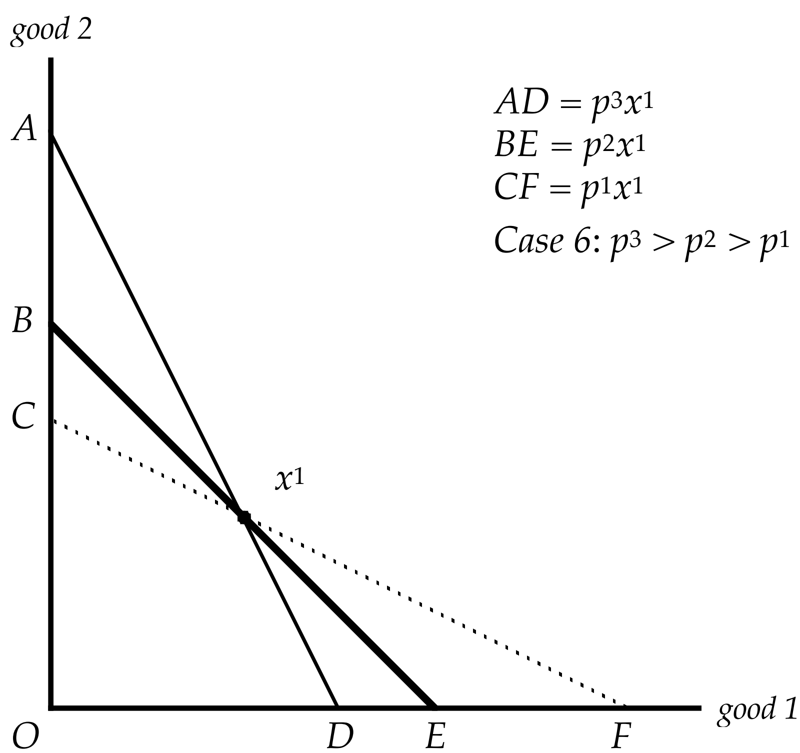

Figure 6